Exercise 5

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = \frac{1}{6}x^3 + \int_0^x (x-t)u(t)\,dt$$

Solution

The successive approximations method, also known as the method of Picard iteration, will be used to solve the integral equation. Consider the iteration scheme,

$$u_{n+1}(x) = \frac{1}{6}x^3 + \int_0^x (x-t)u_n(t)\,dt, \quad n \geq 0,$$

choosing $u_0(x) = 0$. Then

$$\begin{aligned}
u_1(x) &= \frac{1}{6}x^3 + \int_0^x (x-t)u_0(t)\,dt = \frac{1}{6}x^3 \\
u_2(x) &= \frac{1}{6}x^3 + \int_0^x (x-t)u_1(t)\,dt = \frac{1}{6}x^3 + \frac{1}{120}x^5 \\
u_3(x) &= \frac{1}{6}x^3 + \int_0^x (x-t)u_2(t)\,dt = \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 \\
u_4(x) &= \frac{1}{6}x^3 + \int_0^x (x-t)u_3(t)\,dt = \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362\,880}x^9 \\
&\vdots,
\end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=1}^{n+1} \frac{x^{2k+1}}{(2k+1)!}.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned}
\lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \sum_{k=1}^{n+1} \frac{x^{2k+1}}{(2k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} - x \\
&= \sinh x - x
\end{aligned}$$

Therefore, $u(x) = \sinh x - x$.